International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P):2319-3972 ; ISSN(E): 2319-3980
Vol. 5, Issue 1, Dec – Jan 2016; 49-62

International Academy of Science,
Engineering and Technology
Connecting Researchers; Nurturing Innovations

A MULTIVARIATE EXTENSION OF SHAPIRO-WILK'S TEST AND POWER INVESTIGATION FOR MIXED ALTERNATIVES

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ABSTRACT

It is well known that even though many procedures are available for testing univariate normality, the procedure developed by Shapiro and Wilk (1965) is a very effective and powerful test to detect a variety of departures from normality. A generalization of the Shapiro-Wilk procedure to test for multivariate normality has been given recently by Alva and Estrada (2009). The present paper considers a different approach to extend the Shapiro-Wilk procedure for testing multivariate normality. An extensive simulation has been carried out to generate the critical values of the proposed statistic for dimensions 2 and 5. Power comparison of the new approach to the one given by Alva and Estrada (2009) is presented for a choice of mixed alternatives. The software for this work is developed in R Language.

KEYWORDS: Testing Multivariate Normality, Mixtures of Distributions, Monte Carlo Simulation

1. INTRODUCTION

In most of the applications of statistics dealing with single dimension datasets, the univariate normal distribution plays a pivotal role. The information age, with availability of facilities to collect and store large multi-dimension datasets, has propelled corresponding theoretical developments to handle multivariate data analysis. Much of the theoretical developments that have taken place so far are based on the multivariate normal distribution. Thus, many of the procedures for analyzing multivariate response data require joint normality of the multivariate responses. Simulation studies carried out by Mardia (1975) and many others emphasize the importance of the multivariate normality assumption for many of these procedures, showing that they are not robust to non-multivariate-normal data.

A number of procedures to test for multivariate normality exist in the literature. It is found that there are about 40 to 50 different approaches to this testing problem, but no single procedure has been established to be the best to detect various types of departures from multivariate normality. Some of the earliest works in this context are those of Healy (1968) who proposed a plotting technique and Andrews *et al.* (1971) who gave a procedure based on the distribution of Mahalanobis distances. A powerful procedure based on the Shapiro-Wilk (1965) procedure was developed by Royston (1983). Recent contributions in this area include the articles by Szekely and Rizzo (2005), Holgersson (2006), Alva and Estrada (2009) and Tenreiro (2011) among others. Despite the existence of so many tests for multivariate normality, the assumption of multivariate normality frequently goes untested. The reluctance to test for multivariate normality is due to lack of awareness and limited availability of software for many of the existing techniques.

In this paper, we focus on extending the Shapiro-Wilk procedure used to test for univariate normality because this test has been found to be a very powerful one compared to so many other procedures. The recent work of Alva and Estrada

(2009) provides one generalization of the Shapiro-Wilk procedure to test for multivariate normality. These authors investigated the power of their test against a wide variety of alternatives for dimensions $p = 2$ and $p = 5$. Their test performs well for certain alternatives and not so well for others.

The present work considers a different and logically more appealing approach in extending the Shapiro-Wilk procedure to test for multivariate normality. For now, the investigations encompass dimensions 2 and 5 and aim at detecting certain types of mixed alternatives. As there are no algebraic closed-form expressions for the distribution of the Shapiro-Wilk statistic, this study depends on Monte Carlo simulations to develop tables of critical values and to carry out power investigations as has been done by Alva and Estrada (2009).

This paper is organized as follows: In Section 2, apart from giving the basic ideas of Shapiro-Wilk (1965), the approach of Alva and Estrada (2009) is briefly discussed. The modification proposed to the Shapiro-Wilk's approach for multivariate context is presented in Section 3. The critical values of the test statistic for dimensions 2 and 5 are presented in Section 4. Section 5 discusses the results of the power investigations carried out for a class of mixed alternatives and compares the powers with those obtained in the approach of Alva and Estrada (2009). Section 6 contains concluding remarks and highlights the findings of the present investigation. The software for this research has been developed in R language.

2. A REVIEW OF SHAPIRO-WILK AND ALVA-ESTRADA PROCEDURES

Shapiro-Wilk's statistic for testing the hypothesis of univariate normality using a random sample $x_1, x_2 \ldots x_n$ is given by

$$W_X = \frac{S_1^2}{S_X^2} \tag{1}$$

Here $S_X^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$ and $S_1^2 = \left[\sum_{i=1}^{n} a_i\, x_{(i)}\right]^2$, where $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ are the ordered observations and a_i is the i^{th} element of $\boldsymbol{a} = \frac{\underset{\sim}{m}' V^{-1}}{\left[\underset{\sim}{m}' V^{-1} V^{-1} \underset{\sim}{m}\right]^{1/2}}$ with $\underset{\sim}{m}'$ the mean vector and V the var-cov matrix of the vector of order statistics of a random sample of size n from a standard normal distribution. Clearly, $\boldsymbol{a}$ is determined when $\underset{\sim}{m}$ and V are known. However, the elements of V are known only up to sample size 20. And for sample sizes varying from 20 to 50, approximate values were found by Shapiro and Wilk (1965). For higher sample sizes, Royston (1992) provided an approximation of $\boldsymbol{a}$.

The Shapiro-Wilk Test rejects the normality hypothesis with size α if $W_X < k_\alpha$, where k_α is the 100α percentile of the distribution of W_X under the normality hypothesis. Since $\boldsymbol{a}$ is computed numerically and not analytically and as W_X depends on $\boldsymbol{a}$, the distribution of W_X cannot be obtained analytically but only by numerical methods.

Now, we shall discuss the multivariate extension of Shapiro-Wilk procedure given by Alva and Estrada (2009).Let $\mathbf{X_1}, \mathbf{X_2} \ldots \mathbf{X_n}$ be independent and identically distributed random vectors of dimension $p \geq 1$. Denote the p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and var-cov matrix $\boldsymbol{\Sigma}$ by $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Also, let $\mathbf{0}$ be the null vector of order p

and **I** be the identity matrix of order p x p. It is well known that $\mathbf{X_i} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if and only if $\mathbf{Z_i} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X_i} - \boldsymbol{\mu}) \sim N_p(\mathbf{0}, \mathbf{I})$.

Let $\overline{\mathbf{X}} = n^{-1}\sum_{i=1}^{n}\mathbf{X}_i$ be the sample mean vector and $\mathbf{S} = n^{-1}\sum_{i=1}^{n}(\mathbf{X}_i - \overline{\mathbf{X}})(\mathbf{X}_i - \overline{\mathbf{X}})'$ be the sample var-cov matrix and $\mathbf{S}^{-1/2}$ be the positive definite square root of $\mathbf{S}^{-1}$. When $\mathbf{X_1}, \mathbf{X_2} \ldots \mathbf{X_n}$ follow $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the random vectors $\mathbf{Z_i} = \mathbf{S}^{-1/2}(\mathbf{X_i} - \overline{\mathbf{X}})$ have a distribution close to $N_p(\mathbf{0}, \mathbf{I})$. This also means that the coordinates of $\mathbf{Z_i}$ denoted $Z_{1i}, Z_{2i}, \ldots, Z_{pi}$ are approximately independent univariate normal random variables. To test the null hypothesis H_0: $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n}$ is a sample from p-variate normal distribution, Alva and Estrada (2009) proposed the test statistic

$$W_{ave} = p^{-1}\sum_{i=1}^{p} W_{Zi} \tag{2}$$

Where W_{Zi} is the Shapiro-Wilk's statistic evaluated on the i^{th} coordinate of the new coordinate system $\mathbf{Z_i}$. If H_0 were true, each of the W_{Zi} is expected to be close to unity. Alva and Estrada (2009) therefore, proceed with the understanding that in such a case, W_{ave} also is expected to be close to unity. Thus, the test would reject H_0 for 'small' values of the statistic W_{ave}. That is, the test rejects H_0 if $W_{ave} < c_{\alpha;n,p}$ where $c_{\alpha;n,p}$ is a quantity satisfying the condition

$$P\{W_{ave} < c_{\alpha;n,p} \mid H_0\} = \alpha$$

α being the desired size of the test. It is noted that for $p = 1$, the Alva-Estrada statistic reduces to the Shapiro-Wilk statistic.

3. THE MODIFIED EXTENSION OF SHAPIRO-WILK PROCEDURE

The present work is an attempt to develop a modified extension of Shapiro-Wilk procedure compared to that of Alva and Estrada (2009). The procedure given by these authors is based on the 'Average' of the Shapiro-Wilk statistics computed in the 'p' dimensions. However, the averaging of the 'p' statistics may probably 'hide' the presence of a 'small' value among them, leading to the incorrect acceptance of joint normality. The proposed modification is expected to be more sensitive in detecting departures from normality. The statistic that is proposed now is

$$W_{min} = \underset{1 \le i \le p}{Min}\left\{W_{Zi}\right\} \tag{3}$$

If H_0 were true, 'each' of the W_{Zi} is expected to be close to unity; equivalently, the least of the W_{Zi} is expected so. Thus, the test would reject H_0 for 'small' values of the statistic W_{min}. That is, the test rejects H_0 if $W_{min} < d_{\alpha;n,p}$ where $d_{\alpha;n,p}$ is a quantity satisfying the condition

$$P\{W_{min} < d_{\alpha;n,p} \mid H_0\} = \alpha$$

α being the desired size of the test. It is noted that for $p = 1$, the statistic reduces to the Shapiro-Wilk statistic. As was done by Alva and Estrada for the statistic W_{ave}, the null distribution of the statistic W_{min} is obtained by simulation whose results are given in the next section.

4. QUANTILES OF THE NULL DISTRIBUTION OF W_{min}

For dimensions p = 2 and 5 and for sample sizes n = 5, 10, 20 . . . 100 we repeat the sampling 100000 times to generate the null distributions of the statistic W_{min} for each combination.

Table 1: Critical Values $d_{0.05;\, n,\, p}$ of W_{min} and $c_{0.05;\, n,\, p}$ W_{ave}

n	W_{min}(p=2)	W_{ave}(p=2)	W_{min}(p=5)	W_{ave}(p=5)
3	0.7609	0.8159	--	--
4	0.727	0.8077	--	--
5	0.7434	0.8169	--	--
6	0.7592	0.8285	0.7203	0.8598
7	0.7783	0.8411	0.7387	0.869
8	0.7943	0.8521	0.7555	0.8773
9	0.8071	0.8611	0.7721	0.8851
10	0.8188	0.8696	0.7858	0.8918
11	0.8304	0.8774	0.7979	0.898
12	0.8403	0.8843	0.8095	0.9036
13	0.8476	0.8899	0.8189	0.9084
14	0.8559	0.8956	0.8286	0.9127
15	0.8626	0.9003	0.8366	0.9167
16	0.8685	0.9046	0.8439	0.9204
17	0.8748	0.9092	0.8503	0.9237
18	0.8799	0.9123	0.857	0.9266
19	0.8847	0.9161	0.8617	0.9291
20	0.8891	0.9194	0.8673	0.932
30	0.9189	0.9406	0.9035	0.9499
40	0.9359	0.953	0.9239	0.9601
50	0.9468	0.9608	0.9372	0.9668
60	0.9545	0.9663	0.9461	0.9715
70	0.96	0.9704	0.9528	0.9749
80	0.9644	0.9736	0.9579	0.9776
90	0.9679	0.9761	0.9621	0.9798
100	0.9709	0.9783	0.9656	0.9815

Tracking the important quintiles identified over the various stages of repetitions, it was found that the ‘convergence’ happens in a convincing manner even as we reached 10000 repetitions. However, as only the quintiles of order 0.05 namely $d_{0.05;\, n,\, p}$ and $c_{0.05;\, n,\, p}$ are required to carry out the test of size 0.05 for multivariate normality, these alone are summarized in Table 1. We emphasize that, the sample size 'n' is required to be greater than the dimension 'p' to carry out the test and therefore, the cells corresponding to n = 3, 4, 5 are empty when p = 5.

5. POWER INVESTIGATION FOR MIXED ALTERNATIVES

We study the powers of W_{min} and W_{ave} by using Monte Carlo simulation of samples of sizes n = 3 (1) 20 (10) 100 for dimension p = 2 and n = 6 (1) 20 (10) 100 for dimension p = 5 for test size $\alpha = 0.05$. For each (n, p), 10000 random

samples were simulated from each of the specified alternative distribution. The alternatives that have been considered in this paper include mixtures of multivariate normal distributions. The findings are presented in the sequel.

Power Investigations for Dimension p = 2

In this section, we consider the following mixtures of bivariate normal distributions as the alternative distributions:

(1) $(1/3)*N_2(\mathbf{0}, \mathbf{I}) + (2/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_1)$

(2) $(2/3)*N_2(\mathbf{0}, \mathbf{I}) + (1/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_1)$

(3) $(1/3)*N_2(\mathbf{0}, \mathbf{I}) + (2/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{1/2})$

(4) $(2/3)* N_2(\mathbf{0}, \mathbf{I}) + (1/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{1/2})$

(5) $(1/3)*N_2(\mathbf{0}, \mathbf{I}) + (2/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{-1/2})$

(6) $(2/3)*N_2(\mathbf{0}, \mathbf{I}) + (1/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{-1/2})$

(7) $(1/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{1/2}) + (2/3)*N_2(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{1/2})$

(8) $(2/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{1/2}) + (1/3)*N_2(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{1/2})$

(9) $(1/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{1/2}) + (2/3))*N_2(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{1/2})$

(10) $(2/3)*N_2(\mathbf{0}, \boldsymbol{\Sigma}_{1/2}) + (1/3))*N_2(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{1/2})$

Here, the mean vectors considered are:

$\mathbf{0} = (0, 0)'$, $\boldsymbol{\mu}_1 = (3, 3)'$, $\boldsymbol{\mu}_2 = (3, -1)'$

The variance-covariance matrices are:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{\Sigma}_1 = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}, \boldsymbol{\Sigma}_{1/2} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}, \boldsymbol{\Sigma}_{-1/2} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

Here $\boldsymbol{\Sigma}_1$ was considered by Alva and Estrada (2009). The empirical powers of the procedures based on the proposed W_{min} statistic and the existing W_{ave} Statistic of Alva and Estrada (2009) are both reported for the different sample sizes, for the first five alternatives in Table 2 followed by the power curves as functions of the sample size in Figures (1) to (5). The same computations and the power curves for the other five alternatives are given in Table 3 and Figures (6) to (10).

Table 2: Empirical Powers of W_{min} and W_{ave} for p = 2 and Test Size 0.05 for Alternatives (1) to (5)

n	W_{min} (1)	W_{ave} (1)	W_{min} (2)	W_{ave} (2)	W_{min} (3)	W_{ave} (3)	W_{min} (4)	W_{ave} (4)	W_{min} (5)	W_{ave} (5)
3	0.0443	0	0.0481	0	0.0468	0	0.0461	0	0.0492	0
4	0.0449	0.0345	0.0451	0.0352	0.0514	0.0377	0.0501	0.04	0.0507	0.0394
5	0.0431	0.0397	0.0468	0.0425	0.0502	0.0412	0.0486	0.0411	0.046	0.0428
6	0.0437	0.0466	0.044	0.0411	0.0477	0.045	0.0495	0.0472	0.0479	0.0462
7	0.0458	0.0474	0.046	0.0467	0.0447	0.0398	0.0483	0.0446	0.0502	0.0465
8	0.0468	0.048	0.0511	0.05	0.0506	0.0487	0.0468	0.0444	0.043	0.045
9	0.0515	0.0514	0.046	0.0471	0.0492	0.0471	0.0523	0.0523	0.0484	0.0502
10	0.0463	0.0481	0.0429	0.0425	0.0496	0.0503	0.0475	0.0449	0.0483	0.0488
11	0.0543	0.0562	0.0473	0.0486	0.0484	0.0492	0.0502	0.0478	0.0512	0.0488
12	0.0543	0.0532	0.0437	0.0431	0.0496	0.0523	0.0509	0.0514	0.0528	0.0499
13	0.0535	0.0543	0.0447	0.0471	0.0484	0.0491	0.0454	0.0473	0.0467	0.0462
14	0.0561	0.056	0.0468	0.0505	0.0517	0.0515	0.0474	0.0504	0.0477	0.0489

15	0.0601	0.0586	0.0475	0.0472	0.0487	0.0493	0.049	0.0478	0.0531	0.0481
16	0.0642	0.0602	0.0457	0.0494	0.0491	0.0477	0.0519	0.0513	0.0493	0.0497
17	0.0622	0.0641	0.0478	0.0483	0.0471	0.0492	0.0468	0.047	0.05	0.0485
18	0.0638	0.0607	0.0488	0.0518	0.0492	0.0501	0.0505	0.0493	0.046	0.0459
19	0.0608	0.0636	0.0467	0.0477	0.0499	0.0505	0.052	0.0528	0.0501	0.0472
20	0.0631	0.0672	0.0474	0.0497	0.0512	0.052	0.0514	0.0532	0.0497	0.0501
30	0.081	0.0816	0.0495	0.0502	0.0468	0.0448	0.0493	0.0466	0.0523	0.0492
40	0.0897	0.0882	0.0543	0.0553	0.0489	0.0499	0.0501	0.0512	0.0487	0.0492
50	0.1034	0.1057	0.0542	0.0547	0.0498	0.0503	0.05	0.0522	0.052	0.0519
60	0.1135	0.1107	0.0553	0.0546	0.0539	0.0523	0.0489	0.0471	0.0456	0.0455
70	0.1165	0.118	0.0565	0.0559	0.0502	0.0492	0.049	0.0481	0.0481	0.0497
80	0.122	0.1213	0.052	0.0516	0.0497	0.0483	0.0493	0.0491	0.0514	0.0499
90	0.1368	0.1381	0.0544	0.0547	0.0529	0.0526	0.0504	0.0479	0.0525	0.0505
100	0.1415	0.1451	0.0543	0.0581	0.0497	0.0503	0.0508	0.0506	0.0517	0.0518

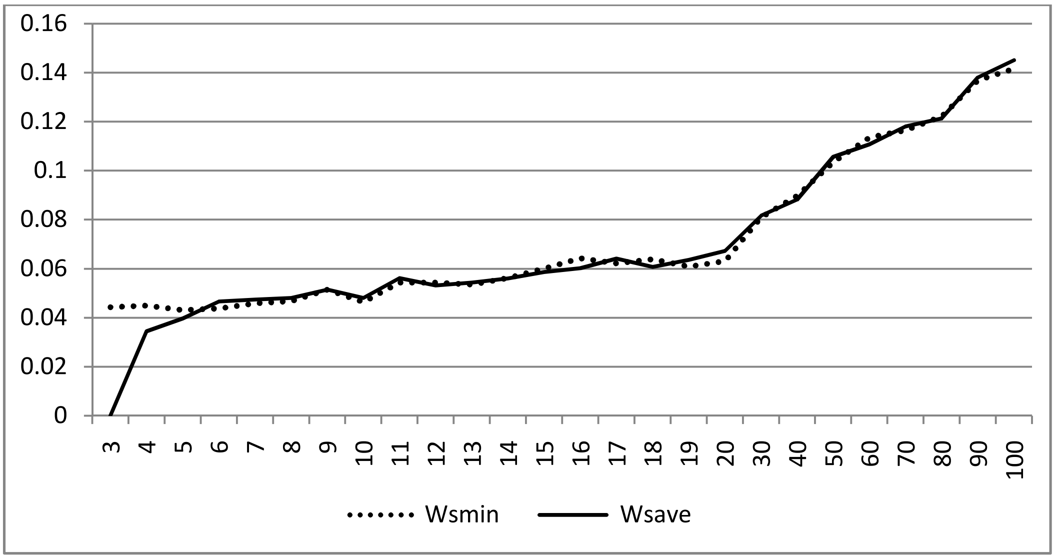

Figure 1: Power Curves for the Mixture Alternative (1/3)*N_2 (0, I) + (2/3)*N_2 (0, Σ_1)

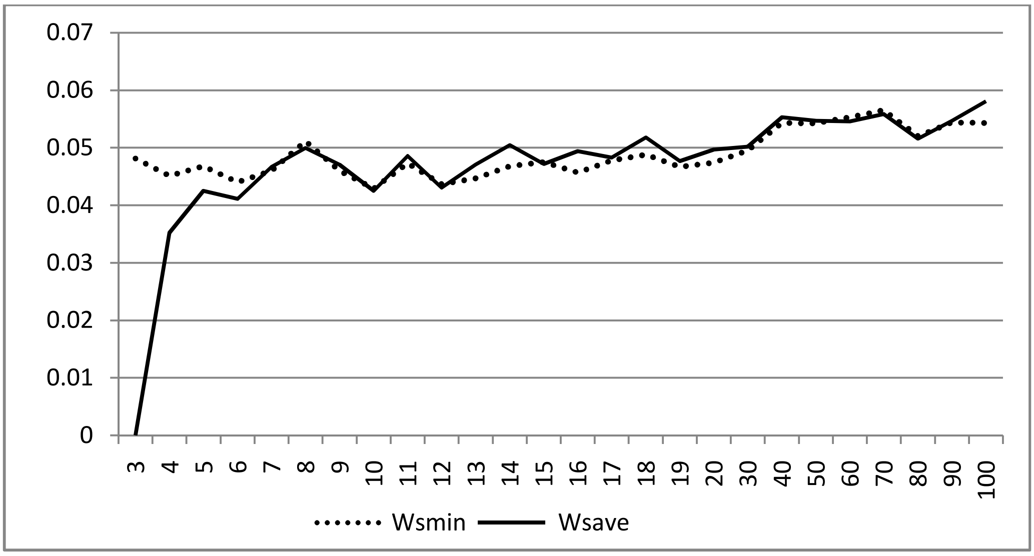

Figure 2: Power Curves for the Mixture Alternative (2/3)*N_2 (0, I) + (1/3)*N_2 (0, Σ_1)

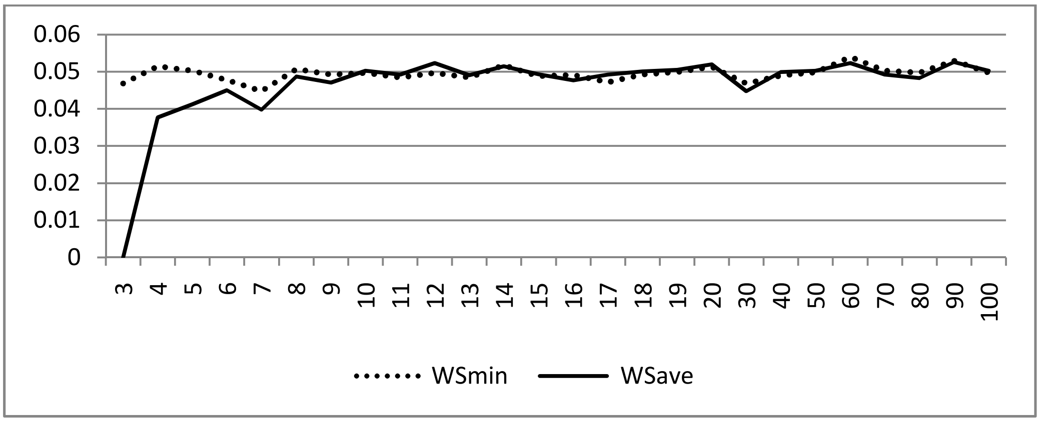

Figure 3: Power Curves for the Mixture Alternative (1/3)*N_2 (0, I) + (2/3)*N_2 (0, $\Sigma_{1/2}$)

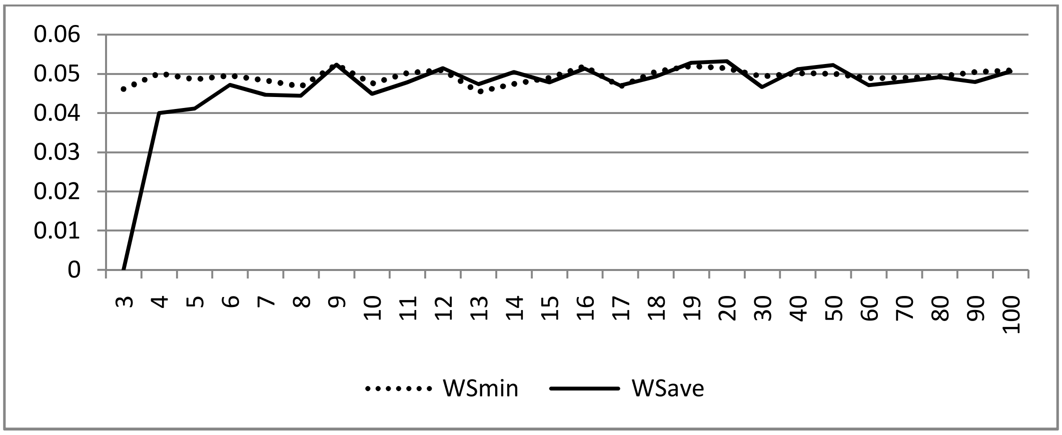

Figure 4: Power Curves for the Mixture Alternative (2/3)*N_2 (0, I) + (1/3)*N_2 (0, $\Sigma_{1/2}$)

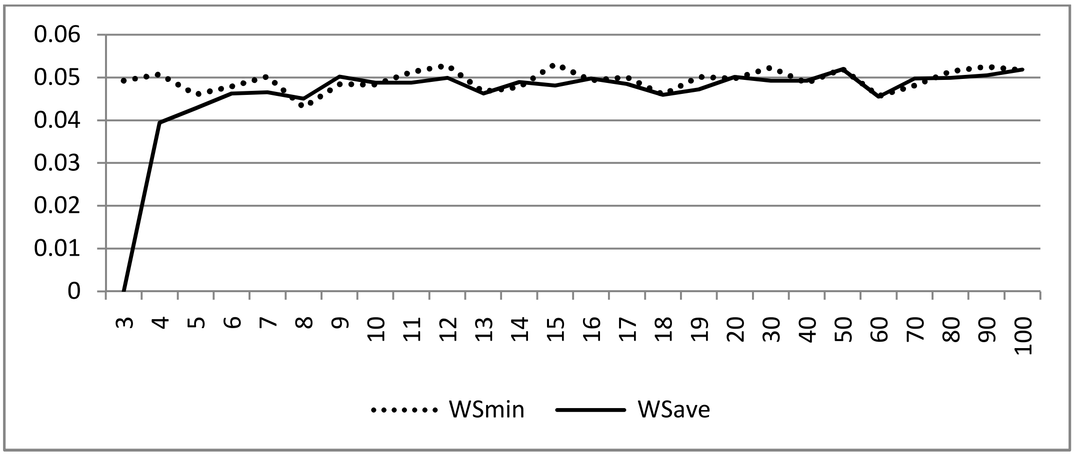

Figure 5: Power Curves for the Mixture Alternative (1/3)*N_2 (0, I) + (2/3)*N_2 (0, $\Sigma_{-1/2}$)

Table 3: Empirical Powers of W_{min} and W_{ave} for p = 2 and Test Size 0.05 for Alternatives (6) to (10)

n	W_{min} (6)	W_{ave} (6)	W_{min} (7)	W_{ave} (7)	W_{min} (8)	W_{ave} (8)	W_{min} (9)	W_{ave} (9)	W_{min}(10)	W_{ave} (10)
3	0.0449	0	0.0481	0	0.047	0	0.0474	0	0.047	0
4	0.0492	0.0392	0.0712	0.0598	0.0336	0.0255	0.0418	0.0437	0.0613	0.0445
5	0.052	0.0451	0.0544	0.0502	0.0347	0.0311	0.0394	0.0432	0.0722	0.0674
6	0.0487	0.0459	0.0477	0.042	0.037	0.0364	0.0437	0.0479	0.0899	0.0917
7	0.0528	0.051	0.043	0.0462	0.0451	0.0468	0.0404	0.049	0.0556	0.0623

8	0.0485	0.0497	0.0434	0.0465	0.0445	0.0477	0.0496	0.0567	0.0774	0.0814
9	0.0473	0.0509	0.0408	0.0426	0.0453	0.0447	0.0492	0.0594	0.086	0.0911
10	0.0452	0.0463	0.0374	0.0418	0.0437	0.0428	0.041	0.0526	0.0614	0.0699
11	0.0505	0.0496	0.0422	0.0452	0.0423	0.0448	0.0518	0.0581	0.0815	0.0855
12	0.0528	0.0532	0.0422	0.045	0.0481	0.049	0.0628	0.071	0.0953	0.1028
13	0.0473	0.048	0.0389	0.0419	0.0423	0.0436	0.0512	0.0629	0.0725	0.0825
14	0.0484	0.0466	0.0393	0.0434	0.0454	0.0472	0.0604	0.0696	0.0882	0.0944
15	0.043	0.0432	0.041	0.0449	0.0491	0.0526	0.0722	0.0806	0.1033	0.111
16	0.0487	0.0497	0.0397	0.0425	0.0406	0.0424	0.0593	0.0754	0.0829	0.0916
17	0.0469	0.0511	0.0408	0.0427	0.0466	0.05	0.0725	0.0821	0.1014	0.1082
18	0.0492	0.049	0.0399	0.0409	0.045	0.0474	0.0839	0.0908	0.1184	0.1201
19	0.0455	0.048	0.0399	0.0411	0.0439	0.0447	0.0717	0.0879	0.1001	0.1077
20	0.0485	0.0481	0.0367	0.0412	0.0486	0.0532	0.0844	0.1002	0.1125	0.1215
30	0.0479	0.0476	0.0443	0.0502	0.0467	0.0518	0.1425	0.1469	0.1855	0.1866
40	0.0471	0.0495	0.0447	0.0515	0.0483	0.0516	0.1935	0.1915	0.2245	0.2213
50	0.0475	0.046	0.0478	0.0519	0.0507	0.0588	0.2767	0.2653	0.3068	0.2928
60	0.046	0.0484	0.0562	0.0624	0.0641	0.0695	0.3596	0.3393	0.3882	0.3658
70	0.0522	0.0504	0.0548	0.0624	0.0635	0.069	0.4224	0.3879	0.4445	0.4121
80	0.0522	0.0523	0.0662	0.0762	0.0663	0.0707	0.5023	0.4632	0.5261	0.4838
90	0.0501	0.0476	0.0721	0.078	0.0727	0.085	0.5803	0.529	0.6003	0.5527
100	0.0527	0.0499	0.0844	0.0923	0.0805	0.0871	0.6609	0.608	0.6592	0.6056

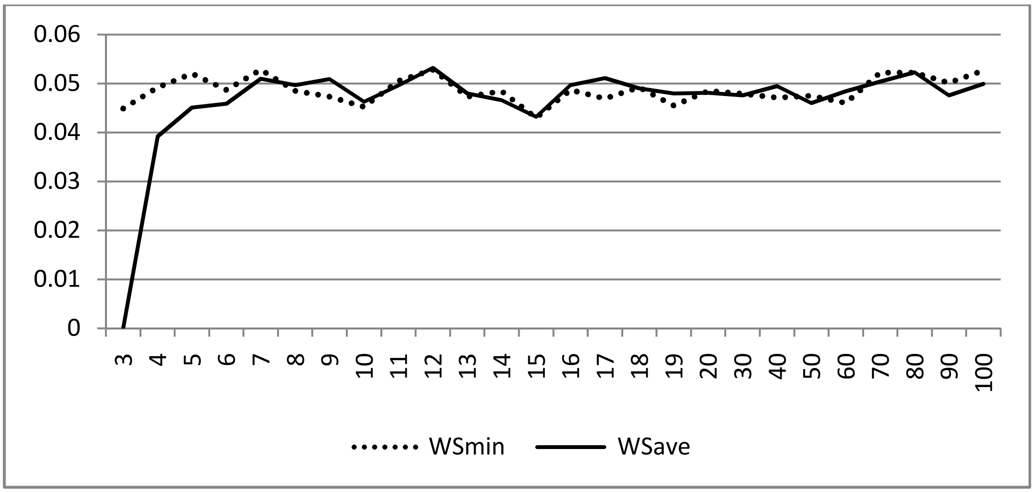

Figure 6: Power Curves for the Mixture Alternative (2/3)*N_2 (0, I) + (1/3)*N_2 (0, $\Sigma_{-1/2}$)

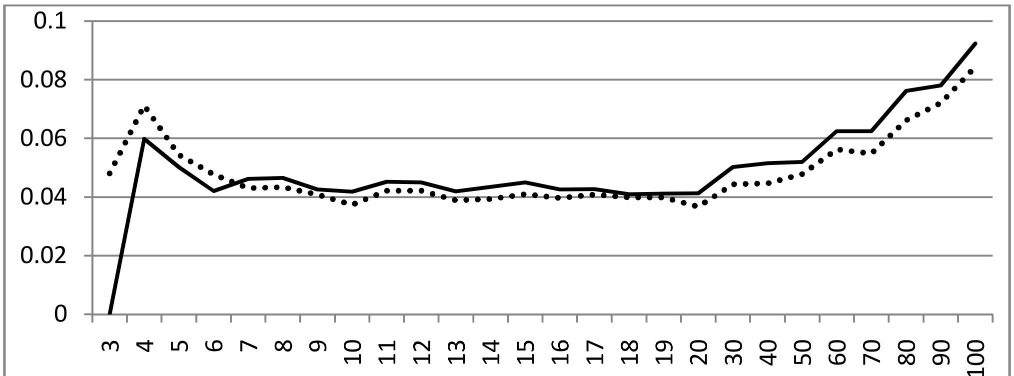

Figure 7: Power Curves for the Mixture Alternative (1/3)*N_2 (0, $\Sigma_{1/2}$) + (2/3)*N_2 (μ_1, $\Sigma_{1/2}$)

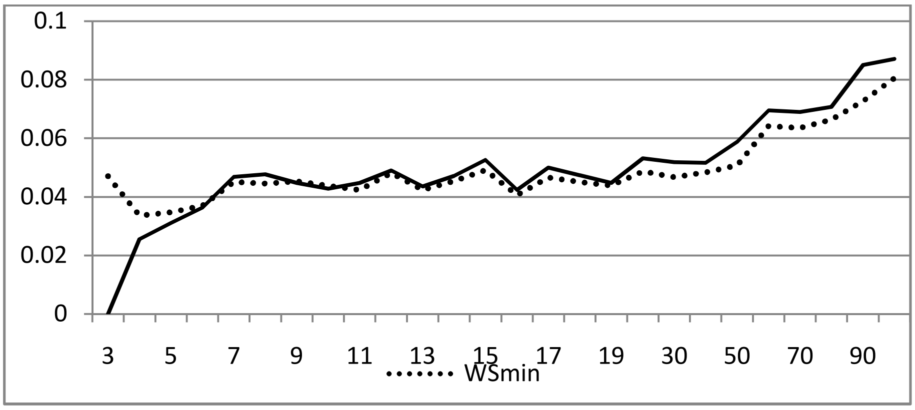

Figure 8: Power Curves for the Mixture Alternative (2/3)*N_2 (0, $\Sigma_{1/2}$) + (1/3)*N_2 (μ_1, $\Sigma_{1/2}$)

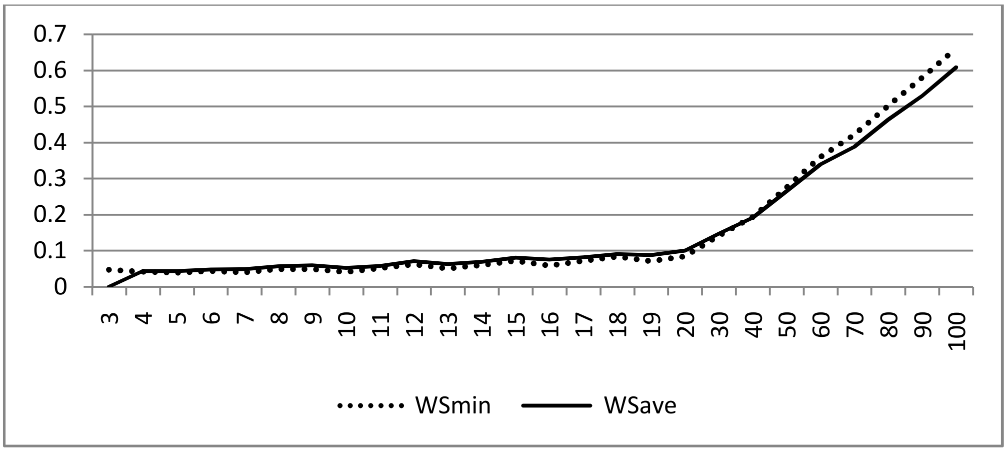

Figure 9: Power Curves for the Mixture Alternative (1/3)*N_2 (0, $\Sigma_{1/2}$) + (2/3))*N_2 (μ_2, $\Sigma_{1/2}$)

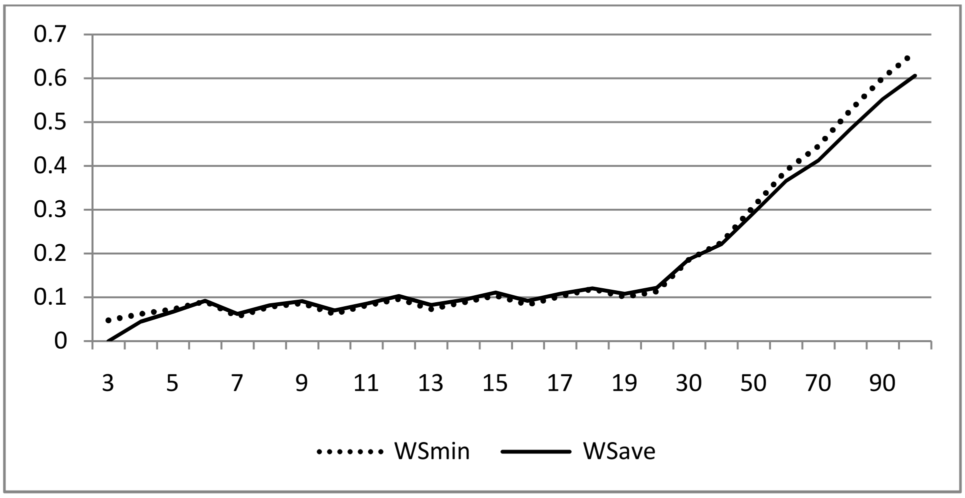

Figure 10: Power Curves for the Mixture Alternative (2/3)*N_2 (0, $\Sigma_{1/2}$) + (1/3))*N_2 (μ_2, $\Sigma_{1/2}$)

Power Investigations for Dimension p = 5

The alternatives considered in this section are the following mixtures of five-variate normal distributions:

(11) (1/3)*N_5 (0, I) + (2/3)*N_5 (0, Σ_2)

(12) (2/3)*N_5 (0, I) + (1/3)*N_5 (0, Σ_2)

(13) (1/3)*N_5 (0, I) + (2/3)*N_5 (0, Σ_3)

(14) (2/3)*N_5 (0, I) + (1/3)*N_5 (0, Σ_3)

(15) (1/3)*N_5 (0, I) + (2/3)*N_5 (0, Σ_4)

(16) (2/3)*N_5 (0, I) + (1/3)*N_5 (0, Σ_4)

Here, the mean vector considered is:

0 = (0, 0, 0, 0, 0)'

The variance-covariance matrices other than identity matrix **I** of order 5 are:

$$\Sigma_2 = \begin{bmatrix} ? & ? & ? & ? & ? \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \Sigma_3 = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}, \Sigma_4 = \begin{bmatrix} ? & ? & ? & ? & ? \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The empirical powers of the procedures based on the proposed W_{min} statistic and the existing W_{ave} Statistic of Alva and Estrada (2009) are both reported for the different sample sizes and for the alternatives (11) to (13) in Table 4 followed by the power curves as functions of the sample size in Figures (11) to (13). The powers for the rest of the alternatives (14) to (16) and their power curves are given in Table 5 and Figures (14) to (16).

Table 4: Empirical Powers of W_{min} and W_{ave} for p= 5 and Test Size 0.05 for Alternatives (11) to (13)

n	W_{min} (11)	W_{ave} (11)	W_{min} (12)	W_{ave} (12)	W_{min} (13)	W_{ave} (13)
6	0.0431	0.0358	0.045	0.0385	0.0444	0.0365
7	0.0433	0.0422	0.0437	0.0431	0.0463	0.0389
8	0.0475	0.0452	0.0489	0.0465	0.0458	0.0454
9	0.0582	0.0598	0.0484	0.0474	0.0459	0.0469
10	0.0591	0.0602	0.0484	0.0502	0.0494	0.0463
11	0.0747	0.0739	0.049	0.0571	0.0511	0.0506
12	0.0856	0.0897	0.0491	0.0544	0.0466	0.0456
13	0.0931	0.0952	0.0525	0.0582	0.047	0.0453
14	0.1034	0.111	0.0546	0.0582	0.0477	0.0508
15	0.1253	0.1361	0.0538	0.056	0.0476	0.0496
16	0.1255	0.129	0.0612	0.0594	0.0483	0.0491
17	0.1346	0.1433	0.0605	0.0648	0.0512	0.0534
18	0.1609	0.1704	0.0603	0.0618	0.0522	0.0495
19	0.1461	0.1539	0.0639	0.0632	0.0497	0.0521
20	0.1768	0.1865	0.0639	0.0639	0.0462	0.0497
30	0.3024	0.3438	0.0743	0.0721	0.0556	0.0561
40	0.376	0.4555	0.0867	0.0839	0.0625	0.0595
50	0.4964	0.6032	0.0931	0.095	0.0652	0.0621
60	0.5848	0.7277	0.0986	0.0959	0.0698	0.0659
70	0.631	0.7792	0.1066	0.1143	0.0677	0.0672
80	0.7106	0.8583	0.1142	0.1179	0.071	0.0721
90	0.7859	0.9137	0.1134	0.1208	0.0713	0.0703
100	0.8386	0.9444	0.1262	0.131	0.0799	0.0764

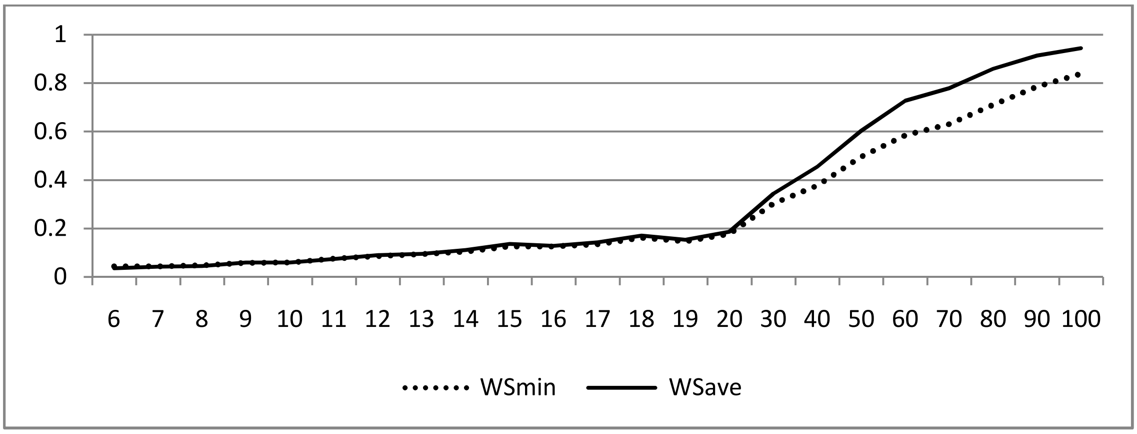

Figure 11: Power Curves for the Mixture Alternative (1/3)*N_5 (0, I) + (2/3)*N_5 (0, Σ_2)

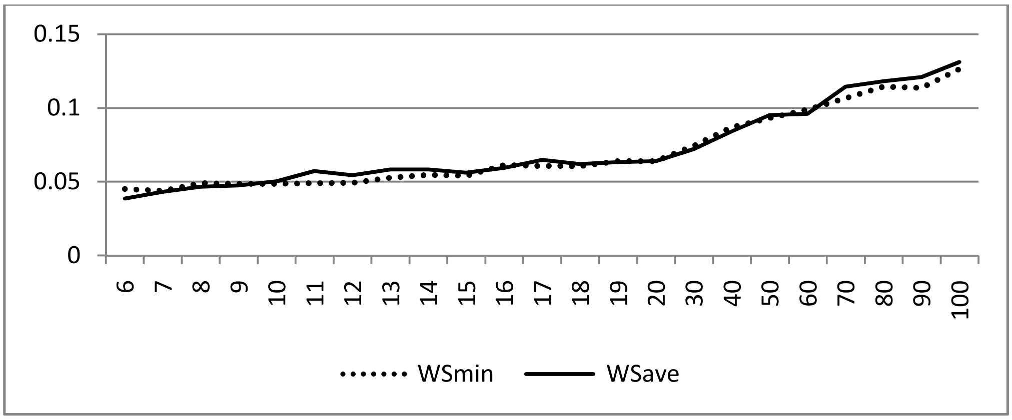

Figure 12: Power Curves for the Mixture Alternative (2/3)*N_5 (0, I) + (1/3)*N_5 (0, Σ_2)

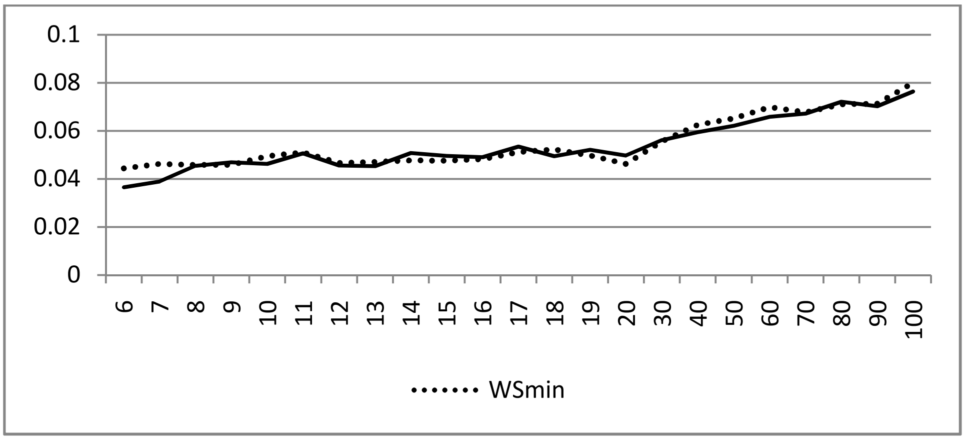

Figure 13: Power Curves for the Mixture Alternative (1/3)*N_5 (0, I) + (2/3)*N_5 (0, Σ_3)

Table 5: Empirical Powers of W_{min} and W_{ave} for p= 5 and Test Size 0.05 for Alternatives (14) to (16)

n	W_{min} (14)	W_{ave} (14)	W_{min} (15)	W_{ave} (15)	W_{min} (16)	W_{ave} (16)
6	0.0487	0.0352	0.0438	0.0384	0.0476	0.0379
7	0.0509	0.0431	0.0472	0.0447	0.047	0.0419
8	0.048	0.0459	0.0469	0.0442	0.0449	0.0423
9	0.0445	0.0448	0.0491	0.0459	0.0461	0.0446
10	0.0469	0.0444	0.0447	0.0454	0.045	0.0468
11	0.0469	0.048	0.0444	0.0459	0.0497	0.052
12	0.0482	0.0471	0.051	0.0483	0.045	0.0489

13	0.0461	0.0469	0.049	0.0493	0.0464	0.0453
14	0.0504	0.0514	0.0474	0.0477	0.0475	0.0487
15	0.048	0.0479	0.0497	0.0498	0.0496	0.05
16	0.0496	0.0486	0.0506	0.0495	0.0463	0.0499
17	0.047	0.047	0.0502	0.0504	0.0464	0.0492
18	0.0485	0.0467	0.0533	0.0528	0.0442	0.0467
19	0.048	0.0489	0.0514	0.0519	0.0504	0.0505
20	0.0489	0.0511	0.0517	0.0546	0.0452	0.0469
30	0.046	0.0446	0.0563	0.0581	0.0477	0.0496
40	0.0537	0.049	0.0628	0.063	0.0548	0.0528
50	0.0543	0.0514	0.0609	0.0627	0.0505	0.0495
60	0.0534	0.0491	0.0717	0.0707	0.055	0.0518
70	0.0528	0.0526	0.0774	0.0751	0.0521	0.0518
80	0.0535	0.0535	0.0687	0.0703	0.0488	0.0527
90	0.0518	0.0564	0.0758	0.0754	0.054	0.0518
100	0.0549	0.0533	0.0789	0.077	0.057	0.054

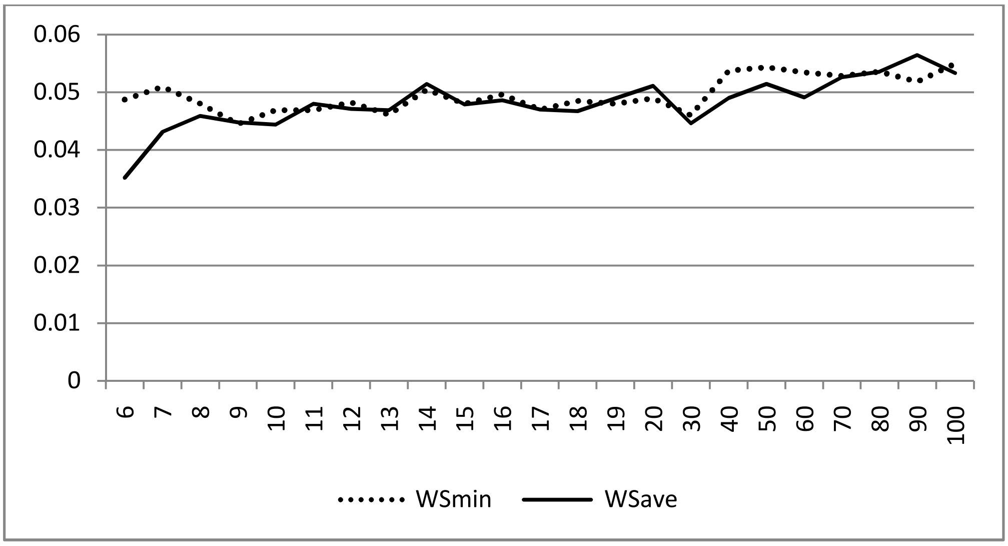

Figure 14: Power Curves for the Mixture Alternative (2/3)*N_5 (0, I) + (1/3)*N_5 (0, Σ_3)

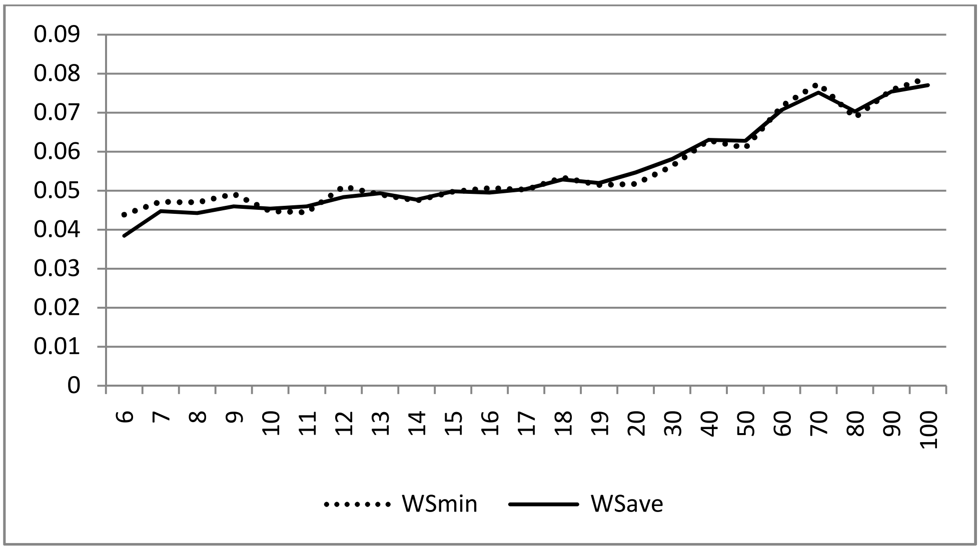

Figure 15: Power Curves for the Mixture Alternative (1/3)*N_5 (0, I) + (2/3)*N_5 (0, Σ_4)

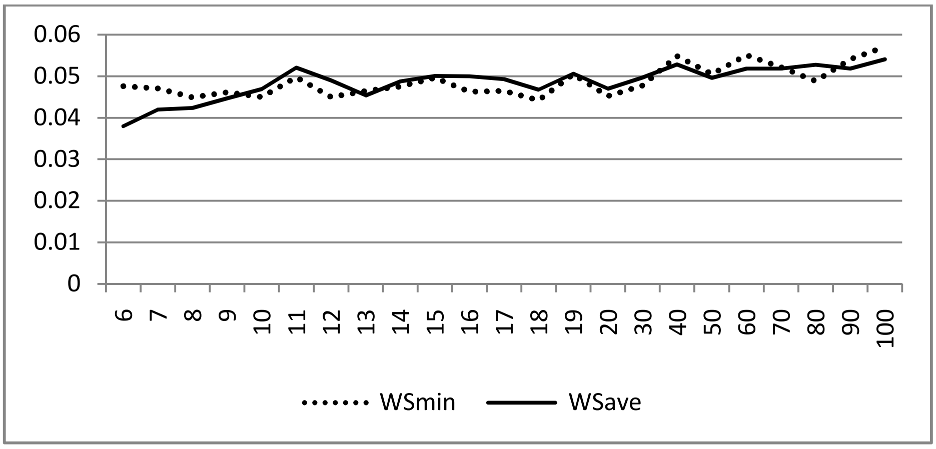

Figure 16: Power Curves for the Mixture Alternative (2/3)*N_5 (0, I) + (1/3)*N_5 (0, Σ_4)

CONCLUSIONS

From the analysis carried out in Section 5 comparing the statistic W_{min} considered in this paper and the statistic W_{ave} considered by Alva and Estrada (2009), it is found that, by and large, W_{min} either outperforms W_{ave} or graciously underperforms. However, the underperformance is not very severe and for whatever sample sizes and alternatives we find W_{min} underperforming than W_{ave}, the difference is not very marked. The perusal of the tables and graphs establishes the fact that the underperformance of W_{min}, whenever it happens, is not severe.

Interestingly, for small sample sizes, we find that W_{min} consistently outperforms W_{ave}. Even in some of the cases where 'closeness' appears between the two for 'small sample sizes', the 'closeness' is due to the 'scaling' done in the vertical axis to accommodate 'large' values of the power function. Such a scenario can be observed in the case of alternative 11 [Refer to Table 4 and Figure 11].

Overall, it is found that, the W_{MIN} procedure looks a promising one in detecting departures from multivariate normality in the form of mixtures of multivariate normal distributions, especially with samples of small sizes. Such a small sample situation involving test for multivariate normality arises in Multivariate ANOVA. However, the test is not carried out by many practitioners due to the absence of a reasonably well-performing procedure for small samples. The approach considered in this paper offers an easy to perform test in such small-sample situations. The performance of this statistic in detecting other types of non-normality in multivariate distributions is under investigation and the results will be communicated in future.

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